**Perturbative Expansion of GF**

**GContour starting in |GS> at t = t0 and Vtime-dependent(t) later**

(remember we’ll be using time-development operator U(t,t0) = U0(t,t0)S(t,t0) here because we’re starting in |GS> at time t0) Can see the single particle file for more details, but as long as the time-dependent parts are switched on after some time we’ll call t = t0, we can work out a self-consistent expansion for GContour averaged against the interacting ground state. The result is:



where the contour starts at -∞, goes past t1,2 to some arbitrary τ, and then back to -∞. And we’ll parenthetically note that this is where we expect the implicit exponential convergence factors attached to the GF’s come from.

**GC starting in |GS> at t = t0 with Vtime-dependent(t) = 0 always**

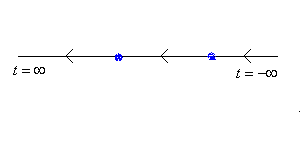
(remember we’ll be using time-development operator U(t,t0) = U0(t,t0)S(t,t0) here because we’re starting in |GS> at time t0) We can obtain GC (t1,t2) already with the formula above, but a simplification is afforded for it in particular if Vtime-dependent(t) = 0, i.e., no time-dependent perturbations.



Then, as we found in the single-particle file,



where the contour now looks as follows:



**Diagrammatic Expansion**

Now let’s consider the diagrammatic expansion of the latter GF. For the sake of discussion we’ll consider a Hamiltonian,



where,



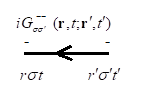
And consider the time ordered green’s function:



Then,

**Real Space Rules**

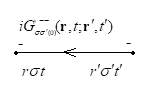
So we have just one GF now



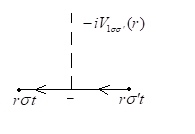
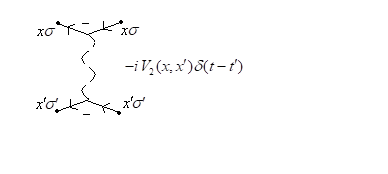
and one set of external points:



which we connect via our unperturbed GF:



and the potential vertices:

Then the general procedure is as follows…

**Topology**

Connect all topologically distinct, fully connected (no bubbles), diagrams together, associating with each element in the diagram the indicated term. See below for justification.

**Equal time issues**

Same as before.

**Signs/Numerical Factors**

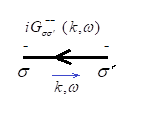
Same as before.

**Sum**

Same as before.

**Fourier Space Rules**

So we have our single GF again:



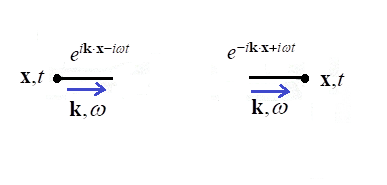
where these G’s are the Fourier transforms of the real space guys,



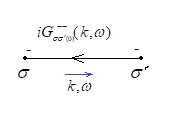
and one set of external points:



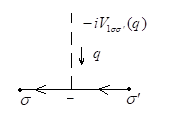
These carry the Fourier transform variable. And if we’re ultimately interested in the real space, time GF, then we must add the following factors to each:



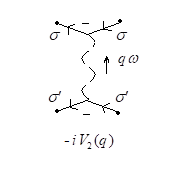
and we connect them with the bare GF



and the single particle potential:



There is no ω-energy flowing down the line because no time dependence. And the two particle potential with (assuming translationally invariant and no time-dependence):



We still have ω flowing down the line because of the artificial time-dependence brought in by the δ(t-t´) we put on each interaction potential.

**Topology**

Connect all topologically distinct, fully connected (meaning no vacuum bubbles), diagrams together, associating with each element in the diagram the indicated term. Remember that all energy-momentum labels **k**, ω must be going in the same way w/r to the GF arrow. They must all be going against the arrows (or with them I suppose). And then we conserve energy/momentum at each vertex. See below for justification…

**Equal time issues**

Apropos the equal time issue, in Fourier space this would be handled as:



So we need to include a factor of exp(iω0+) for each bare Green’s function closed in on itself. Fundamentally, the problem occurs when the temporal argument of the bare Green’s function is zero.

**Signs/Numerical Factors**

Same as above.

**Sum**

Then sum/integrate over all independent momenta - wavenumbers/energies-frequencies/indices. If finite spatial transform, then the guy on the left. If continuous spatial transform, then the guy on the right:



and include the external frequency/wavenumber if want the real space GF.

**Justification of some of these assertions**

Inserting our expression for S into the Green’s function formula, we find



All operators’ time development is implicitly taken to be in the interaction picture, and A, B are presumed also to be in said picture. Recalling the discussion in the 2nd quantization file, especially Wick’s theorem, we see that to evaluate the expectation in the numerator, we can simply decompose it into the sum of all permutations of full contractions (not forgetting to append the appropriate minus sign if necessary to bring the contracted items together – which is what the fermion loop factor is supposed to address). Note that this result applies to any perturbation – not just interactions.

**The -i associated with each potential**

So we can see above that -i and V each occur to the same power. So we can slide the -i next to the potential, and this is where the -i factor comes from.

**The δ(t-t´) associated with the V2 interaction**

Let’s write out the term in more detail,



So we have that we can simplify our rules and associate each end of the interaction vertex with a separate space-time point, if we include the δ function. Of course we’ve also doubled the number of time integrations as well, but the simplicity is worth it I imagine. So now we can write, to the extent that we’re dealing with V2 interactions:

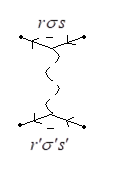


where,



**Elimination of the (1/2) associated with the V2 interaction**

These goes away because there are two equivalent ways to connect to the interaction. We can have one external leg going to rσs and the other to r´σ´s´, or vice versa.



But these are both the same diagram because the interaction is an even function of rσs – r´σ´s´. So we can just connect one way and multiply by two. This is for case when have external legs connected to a diagram.

**That A and B must be topologically connected to each other**

Going to use a bit of notation. First, I’m going to go back to treating each operator as just a single function of time b/c that’s more compact here. And then, since each operator is associated with an integration over its time argument (coming from it being evaluated in the integration picture), I’ll write:



where the operator thing on top is meant to remind us we have to integrate over time. Okay, now note that if A and B are single particle creation/annihilation operators (in no particular order), and because V1 and V2 contain even powers of creation/annihilation operators, there is no way to connect A to one set and B to another, because that would mean A + set 1 would have an odd number of what A is, but an even number of what B is. And B + set 2 would have an odd number of whatever B is, but an even number of whatever A is. And any expectation with a mismatch of powers of ψ and ψ† will be zero. So A has to be connected topologically to B. So we have:



But note that if A and B were say bilinear operators, like we have for phonons and photons, then the two wouldn’t have to be connected. And even regardless, this doesn’t mean that A and B can’t be connected to each other, but disconnected from the rest of the diagram – like vacuum bubbles. This is what the next part is about.

**Elimination of vacuum bubbles**

So we have:



Now we will apply Wick’s theorem to break the many particle expectation into sums of all possible contractions. Looking at the nth order term, we can classify the possible contractions into n distinct sets according to the number m = 0,1,…,n of V’s that are contracted connectedly (because they have to be as argued above) with the external points t, tʹ and (n-m) V’s which are contracted (in no particular way) amongst themselves. Note that the multiplicity of the mth set is equal to:



equal to the number of ways to choose m V’s to fully contract connectedly to the external points, and (n – m) V’s to contract amongst themselves (can have bubbles among the last set). All these permutations within the mth set are all equivalent because we integrate over all the s’s which are just dummy variables. So we have:



(note again, the multiplicity just refers to the number of ways we could get the left < > with the external points, not the number of ways the right < > can be broken up into equivalent terms) or in other words,



We now plug this back into the formula for the Green’s function



Now we want to somehow separate the connected diagrams from bubbles. To this end we’ll reorganize the sum as follows…



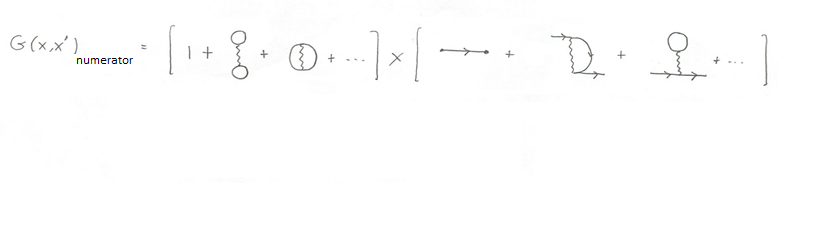
So we have completely factorized the connected and disconnected diagrams, and we can write this as, summing the product



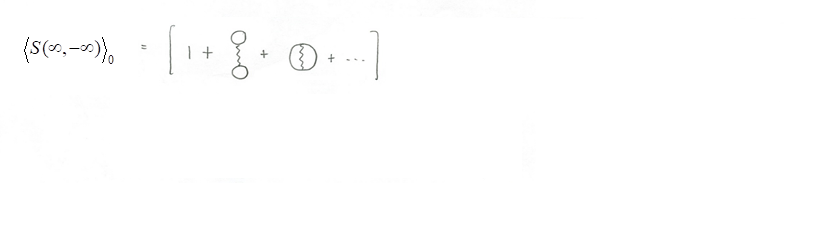
We’ll recognize the right < > as just the denominator of the GF. And so we have:



In pictures, basically we have:



and



Thus the sum of all vacuum bubbles will cancel in the numerator and denominator and we’ll be left simply with the sum of all fully connected diagrams – as we know. So we will see that we can incorportate a self energy into the expression for the Green’s function, as we have been able to do before.

**Elimination of topologically equivalent diagrams (basically the 1/n! factor)**

The next simplification occurs when we realize that



because in the second line there will be n! topologically indistinct diagrams (meaning that the difference in the diagrams consists merely in a relabelization of the internal points). The reason is the following. Suppose that we have a certain full contraction. Then imagine that we leave the contraction lines ‘fixed’ and just permute the order of the V’s, in the same fashion that we do here



(the contraction line stays fixed between the first and third items, we just permute the arrangement of the terms). Then this new contraction which presently would be included as a distinct term in the perturbative expansion will just give us the same value as the unpermuted one because we have effectively only changed the labels on the s’s, and they are dummy variables. Now there are n! ways to permute the positions of the H’s, so the n! topologically indistinct diagrams all give the same value. But note this happens only for diagrams that are fully connected. This doesn’t apply to vacuum bubbles, which instead retain a symmetry factor of 1/p, because our argument here will have over estimated the number of different ways we can construct the bubble. But of course, we have no vacuum bubbles in this GF by the prior argument.

**What about the sign convention?**

We have yet to prove that we get (-1) for each fermion loop. But oh well.

**Special Case V = V1(x)**

Well, you can verify that we get the same expansion for GF as we got from the GF approach.